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Engineering Notes

Cayley Family of Attitude Coordinates

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Introduction

EULER'S rotation theorem has become a guiding principle and starting point for studying the relative orientation of a rigid body. The theorem states that the general displacement of a rigid body with one point fixed is a rotation through some angle about some line [1]. The line, commonly called Euler's principal line, is the eigenvector associated with the $+1$ eigenvalue of the relative orientation matrix $\boldsymbol{C}$. The amount of rotation is called Euler's principal angle. An Euler theorem perspective of orientation considers the unit-length principal line $\boldsymbol{e}$ and principal angle ϕ to be the fundamental components for representing orientation, and results are built upon them.

A different perspective on rotational displacement considers the orientation matrix $\boldsymbol{C}$ as the most fundamental representation for describing relative orientation. From this viewpoint, Euler's theorem becomes an interpretation of relative orientation, or an interpretation of $\boldsymbol{C}$. Recent results show how a direct use of the orientation matrix can lead to new attitude coordinates and new relational expressions [2]. A part of that work was a new Cayley-like transform between so-called interior parameters (IPs) and the orientation matrix $\boldsymbol{C}$. The purpose of this note is to generalize those Cayley-like identities to create a family of attitude coordinates. First, however, an Euler theorem perspective is reviewed.

Euler Family of Attitude Coordinates

It is well-known that Euler's principal line $\boldsymbol{e}$ and angle ϕ can be used to define a family of attitude coordinates via $\boldsymbol{x} = f(\phi)\boldsymbol{e}$. This collection of coordinates can be called an Euler family of attitude coordinates. The scalar function f is assumed to be a smooth function of the principal angle. Although this expression is commonly the extent to which this general relationship is developed, one could define inverse relationships in a straightforward way, viz., $\phi = f^{-1}(x)$ and $\boldsymbol{e} = \boldsymbol{x}/x$, where $x \equiv \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}} = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

The associated kinematic equations needed to study rotational motion can be developed through time differentiation. One is usually interested, however, in functional relationships between the change in the attitude coordinates and the relative angular velocity, notationally written as $\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\omega})$ and $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{x}, \dot{\boldsymbol{x}})$. These kinematic expressions can be effectively determined using the relationship between the angular velocity vector and the time derivative of ϕ and $\boldsymbol{e}$, which Hughes develops in [3].

$$\boldsymbol{\omega} = \dot{\phi}\boldsymbol{e} - (1 - \cos\phi)\boldsymbol{E}\dot{\boldsymbol{e}} + \sin\phi\,\dot{\boldsymbol{e}} \tag{1}$$

$$\dot{\phi} = \boldsymbol{e}^{\mathsf{T}}\boldsymbol{\omega}; \qquad \dot{\boldsymbol{e}} = \tfrac{1}{2}[\boldsymbol{E} - \cot(\phi/2)\boldsymbol{E}\boldsymbol{E}]\boldsymbol{\omega} \tag{2}$$

Here, $\boldsymbol{E}$ is a skew-symmetric matrix arrangement of the elements of the principal line $\boldsymbol{e}$: $e_1 = E_{32}$, $e_2 = E_{13}$, and $e_3 = E_{21}$.

The preceding equations lead to a pair of kinematic expressions involving the attitude coordinates, their rates, and the relative angular velocity vector.

$$\dot{\boldsymbol{x}} = \{(\partial f/\partial\phi)\boldsymbol{x}\boldsymbol{x}^{\mathsf{T}}/x^2 + (1/2)[\boldsymbol{X} - \cot(\phi/2)\boldsymbol{X}\boldsymbol{X}/x]\}\boldsymbol{\omega} \tag{3}$$

$$\boldsymbol{\omega} = \{\boldsymbol{x}\boldsymbol{x}^{\mathsf{T}}/x^2 - [(1-\cos\phi)\boldsymbol{X}/x - \mathbf{1}\sin\phi](\mathbf{1}/x - \boldsymbol{x}\boldsymbol{x}^{\mathsf{T}}/x^3)\}\dot{\boldsymbol{x}} \tag{4}$$

The boldface numeral one denotes the 3×3 identity matrix and $\boldsymbol{X}$ is a skew-symmetric matrix arrangement of the elements of $\boldsymbol{x}$. Also, ϕ and the Jacobian function $\partial f/\partial\phi$, listed here, are understood to be functions of $\boldsymbol{x}$.

Similar to what has been shown here for $\boldsymbol{e}$ and ϕ, a Cayley family of coordinates and the associated kinematics based directly on $\boldsymbol{C}$ can be created. As a first step, some well-known Cayley transform identities are generalized.

Generalized Cayley Transforms

The traditional Cayley transform in n-dimensional space relates an orthogonal orientation matrix $\boldsymbol{C}$ to a skew-symmetric matrix $\boldsymbol{Q}$. The transforms, from $\boldsymbol{C}$ to $\boldsymbol{Q}$ and back again, possess a remarkable symmetry:

$$\boldsymbol{Q} = (\mathbf{1} + \boldsymbol{C})^{-1}(\mathbf{1} - \boldsymbol{C}) = (\mathbf{1} - \boldsymbol{C})(\mathbf{1} + \boldsymbol{C})^{-1} \tag{5}$$

$$\boldsymbol{C} = (\mathbf{1} + \boldsymbol{Q})^{-1}(\mathbf{1} - \boldsymbol{Q}) = (\mathbf{1} - \boldsymbol{Q})(\mathbf{1} + \boldsymbol{Q})^{-1} \tag{6}$$

In three-dimensional space, the three elements of $\boldsymbol{Q}$ are the elements of the classic Rodrigues attitude coordinates or classic Rodrigues parameters (CRPs). Inspired by these relationships, a slight generalization is introduced:

$$\boldsymbol{P} = k(\mathbf{1} + \boldsymbol{C})^{-1}(\mathbf{1} - \boldsymbol{C}) = k(\mathbf{1} - \boldsymbol{C})(\mathbf{1} + \boldsymbol{C})^{-1} \tag{7}$$

$$\boldsymbol{C} = (k\mathbf{1} + \boldsymbol{P})^{-1}(k\mathbf{1} - \boldsymbol{P}) = (k\mathbf{1} - \boldsymbol{P})(k\mathbf{1} + \boldsymbol{P})^{-1} \tag{8}$$

Here, k is an appropriate scalar that will be addressed momentarily.

To confirm these orthogonal and skew-symmetric matrix identities, first note that the parenthetical matrix factors $(\alpha\mathbf{1} - \boldsymbol{M})$ and $(\alpha\mathbf{1} + \boldsymbol{M})$ commute for any square matrix $\boldsymbol{M}$ and scalar α. Consequently, the right-most parts of Eqs. (7) and (8) are true, provided there are no singularity issues. By singularity issues, the following is meant: in the traditional Cayley transforms, k is identically one, and singularity issues arise if $(\mathbf{1} + \boldsymbol{C})$ has one or more zero eigenvalues. The generalized Cayley transform involves a scalar k; however, that may annihilate the zero eigenvalue issue.

It is presented in the Appendix that Eq. (7) generates a skew-symmetric matrix $\boldsymbol{P}$ for a given orthogonal matrix $\boldsymbol{C}$, whereas Eq. (8) generates an orthogonal matrix $\boldsymbol{C}$ for a given skew-symmetric matrix $\boldsymbol{P}$. Furthermore, the connection between Eqs. (7) and (8), which is to say that the matrices $\boldsymbol{P}$ and $\boldsymbol{C}$ in these equations are consistent, is also demonstrated.

The generalized Cayley transforms hold in n-dimensional spaces, but it is difficult in these higher-dimensional settings to imagine guidelines to select a meaningful scalar k. The following section

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Table 1 Attitude coordinates as defined by the appropriate k

$k(\boldsymbol{P})$	$k(\boldsymbol{C})$	Attitude coordinates
1	1	CRPs
$(1-p^2)/2$	$\sqrt{\varsigma+1}/(\sqrt{\varsigma+1}+2)$	IPs
$2+\sqrt{4-p^2}$	$\varsigma+1$	Skew parameters
p^2	$(\varsigma+1)/(3-\varsigma)$	Reciprocal parameters
$(3p^2-1)/(p^2-3)$	$k_{\min}$	Cubic parameters

specializes these generalized Cayley transforms to three-dimensional spaces to introduce new rigid body attitude coordinates.

Cayley Family of Attitude Coordinates

Focusing now on three-dimensional spaces, it is noted that an equivalent representation of the generalized Cayley expressions can be established:

$$\boldsymbol{P} = \frac{k}{\varsigma+1}(\boldsymbol{C}^{\mathsf{T}} - \boldsymbol{C}) \tag{9}$$

$$\boldsymbol{C} = \mathbf{1} - \frac{2k}{p^2+k^2}\boldsymbol{P} + \frac{2}{p^2+k^2}\boldsymbol{PP} \tag{10}$$

Here, ς is the trace of $\boldsymbol{C}$ and $p^2 \equiv \boldsymbol{p}\cdot\boldsymbol{p} = p_1^2 + p_2^2 + p_3^2$. The equivalence of these expressions to Eqs. (7) and (8), respectively, can be demonstrated using the Cayley–Hamilton theorem (as shown in the Appendix).

Equation (9) is a coordinate extraction statement: given an orthogonal orientation matrix $\boldsymbol{C}$, directly compute the corresponding attitude coordinates. Consequently, the scalar k must be a function of the elements of the orientation matrix [i.e., $k(\boldsymbol{C})$]. Conversely, Eq. (10) builds the orthogonal orientation matrix associated with a given set of attitude coordinates. The scalar k must now be a function of the elements of the attitude coordinates [i.e., $k(\boldsymbol{P})$]. Some familiar and new attitude coordinates as defined by the appropriate k are listed in Table 1.

The CRPs are well known to the rigid body attitude community. The IPs, recently introduced in [2], are a natural partition and recollection of the traditional modified Rodrigues parameters and their shadow set (i.e., they are like a restrictive patching, such that the magnitude of the parameters is bounded above by one). For a given orthogonal orientation matrix $\boldsymbol{C}$, Eq. (9) (written in terms of the IPs) will always and automatically give parameters associated with the shortest distance to the origin. The skew parameters, introduced here, are so named because the particular k produces $\boldsymbol{P} = \boldsymbol{C}^{\mathsf{T}} - \boldsymbol{C}$. The reciprocal parameters, also introduced here, are so named because they somewhat resemble the reciprocal of the CRPs [i.e., an element of the reciprocal parameters can be written as $p_i = \tan^{-1}(\phi/2)e_i$]. The usefulness of the skew and reciprocal parameters is suspect, however, and they are included only because the simple forms for k seem promising. The cubic parameters are related to the attitude coordinates introduced by Tsiotras et al., wherein they defined $\boldsymbol{C} = (\mathbf{1}-\boldsymbol{P})^3(\mathbf{1}+\boldsymbol{P})^{-3} = (\mathbf{1}+\boldsymbol{P})^{-3}(\mathbf{1}-\boldsymbol{P})^3$ [4]. These cubic parameters are studied more closely in the next section, and the meaning of $k_{\min}$ is explained.

Cubic Parameters

More insight into the cubic parameters can be gleaned by investigating three relationships: the $k(\boldsymbol{P})$ expression, the cubic polynomial that governs $k(\boldsymbol{C})$, and an expression for the orientation matrix:

$$k = (3p^2-1)/(p^2-3) \Rightarrow p^2 = (3k-1)/(k-3) \tag{11}$$

$$(\varsigma-3)k^3 - 3(\varsigma-3)k^2 + 3(\varsigma+1)k - (\varsigma+1) = 0 \tag{12}$$

$$\boldsymbol{C} = \mathbf{1} - \frac{2(3p^2-1)(p^2-3)}{(p^2+1)^3}\boldsymbol{P} + \frac{2(p^2-3)^2}{(p^2+1)^3}\boldsymbol{PP} \tag{13}$$

First, from Eq. (13), note the triplicity of solutions associated with an orientation at the origin: if $\boldsymbol{C} = \mathbf{1}$, then the parameters satisfy $p^2 = 0$ or $p = \pm\sqrt{3}$. But, $\boldsymbol{C} = \mathbf{1}$ means $\varsigma = 3$; thus, Eq. (12) gives $k = 1/3$, and the right-most expression in Eq. (11) gives $p^2 = 0$. The other solutions at the origin ($p = \pm\sqrt{3}$) produce the unacceptable result $k = \infty$.

There are also issues at 180 deg (note that the angular measures discussed in this note are taken to mean Euler's principal angle). At 180 deg, $\varsigma = -1$ and Eq. (12) gives $k = 0$ or $k = 3$. Using the right-most expression in Eq. (11), the solution $k = 0$ gives $p = \pm 1/\sqrt{3}$, whereas $k = 3$ gives the unacceptable result $p = \infty$.

To help understand these various solutions, the roots of Eq. (12) are computed as a function of ς. Generally, the coefficients that compose this cubic polynomial generate three real and unequal roots (k_1, k_2, and k_3). At 180 deg, however, where $\varsigma = -1$, two roots coalesce at zero. The right-most part of Eq. (11) shows that each root can generate two solutions for the magnitude of the cubic parameters. Direct computation reveals that the root with the smallest absolute value leads to cubic parameters with a magnitude bounded between $\pm 1/\sqrt{3}$. Consequently, a judicious choice for the cubic parameters is $k = k_{\min} \equiv \min(|k_1|, |k_2|, |k_3|)$.

Stereographic projection concepts can provide additional understanding. Recall that stereographic projection is a useful device to visualize the individual elements of the traditional CRPs and IPs as they relate to the attitude quaternion [5,6]. For each of these parameters, an orientation attitude is represented as a point on a two-dimensional unit circle. The idea behind stereographic projection is to map an orientation point on the unit circle to a location on a line by using a single special point. The line is called the projection line, and the special point is called the projection point. The mapping is done by connecting the orientation point to the projection point. The intersection of the resulting connecting line and projection line is the individual parameter value.

Figure 1 shows stereographic projection results for a representative element of the cubic parameter vector. Also shown are graphs for the IPs and CRPs. It is helpful to start with the IPs in explaining this plot.

The IP projection line is labeled as the IP projection curve. All possible orientations, from 0 to ± 180 deg, are mapped to this projection line, and the ordinate value along this line represents a single IP value. It was shown in [2] that the IP form for k, $k = \sqrt{\varsigma+1}/(\sqrt{\varsigma+1}+2)$ leads to a global, nonsingular, three-parameter set that is two-valued at orientations corresponding to 180 deg. The parameters exhibit a discontinuity at the 180 deg configuration, and they naturally represent the shortest distance back to the origin, which is given by the minimum angular distance and is

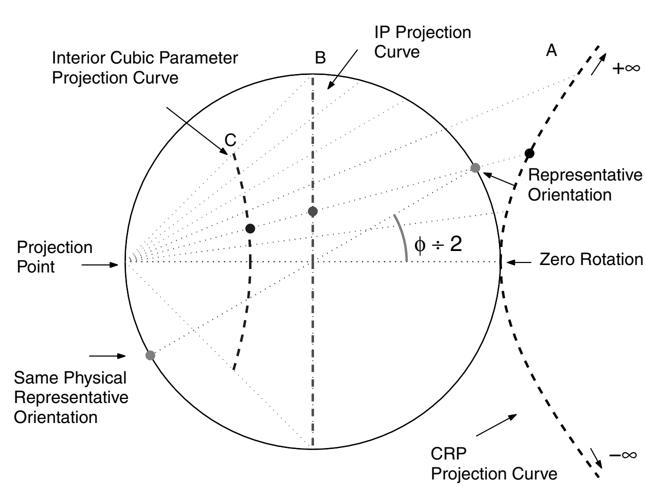

Fig. 1 Stereographic projection results for a representative element of the CRPs (A), the IPs (B), and the I3Ps (C). The parameter values correspond to the ordinate values along the respective projection curves. The set of IPs are double-valued and discontinuous at 180 deg, whereas the CRPs are undefined at 180 deg.

known to be Euler's principal angle. The parameter magnitude is bounded above by one.

Regarding the cubic parameters, the cubic parameter projection line is labeled as the interior cubic parameter (I3P) projection curve, and this is shown as the left-most graph in Fig. 1. All possible orientations, from 0 to ± 180 deg, are mapped to this projection curve, and the ordinate value along this curve represents a single parameter value. Similar to the IPs discussed in the preceding paragraph, when the cubic parameter form for k ($k = k_{\min}$) is used, Eq. (9) produces a global, nonsingular, three-parameter set that is two-valued at orientations corresponding to 180 deg. The parameter magnitude is bounded above by $1/\sqrt{3}$. Similar to the IPs, these parameters exhibit a discontinuity at the 180 deg configuration, and they naturally represent the shortest distance back to the origin. These parameters are appropriately named the I3Ps.

Briefly returning to the IPs, it was shown in [2] that they are related to the second-order attitude coordinates $\boldsymbol{C} = (\mathbf{1} - \boldsymbol{P})^2(\mathbf{1} + \boldsymbol{P})^{-2} = (\mathbf{1} + \boldsymbol{P})^{-2}(\mathbf{1} - \boldsymbol{P})^2$ introduced by Tsiotras et al. [4]. Consequently, in light of the discussion on the cubic parameters, the IPs could be suitably renamed the interior quadratic parameters (I2Ps).

Finally, the CRP projection line is labeled as the CRP projection curve, and this is shown as the right-most graph in Fig. 1. Like before, all possible orientations, from 0 to ± 180 deg, are mapped to this projection curve, and the ordinate value along this curve represents a single parameter value. Notice that the rays corresponding to ± 180 deg are asymptotes of the projection curve, which illustrates the well-known orientation singularity of the CRPs at 180 deg.

Discussion

Key Relationship, More Interior Parameters, and Linearity

Choosing a form for k leads to Cayley-based attitude coordinates similar to the way a choice of f leads to Euler-based attitude coordinates. One expression that plays a critical role is a relationship involving k, orientation, and the attitude coordinates $\varsigma = (3k^2 - p^2)/(k^2 + p^2)$. For example, the choice $k = (p^4 - 6p^2 + 1)/(1 - p^2)/4$ leads to a quartic polynomial that governs $k(\boldsymbol{C})$:

$$(\varsigma - 3)k^4 - 4(\varsigma - 3)(\varsigma + 1)k^3 + 6(\varsigma - 3)(\varsigma + 1)k^2 - 4(\varsigma + 1)^2 k + (\varsigma + 1)^2 = 0 \tag{14}$$

Computation reveals that the root with the smallest absolute value leads to attitude parameters with a magnitude bounded between $\pm\sqrt{3 - 2\sqrt{2}}$. These parameters are the IP version of the fourth-order attitude coordinates introduced in [4], and therefore they are named the interior quartic parameters (I4Ps).

A better comparison of the parameters, perhaps, is shown in Fig. 2. Here, a graph of the I4Ps appears on the far left, and the orientation angle is measured from the projection point. The grid is convenient to read and compare the attitude parameter values associated with an orientation attitude. Interestingly, using a common projection point and collecting the parameter graphs in a common plot beautifully illustrates the more linear characteristics of higher-order attitude parameters that are commonly mentioned [4]. For example, from the graphs in Fig. 2, one can see that the I2Ps are a more linear function of Euler's principle angle than the CRPs, whereas the I3Ps are more linear than the I2Ps, and the I4Ps are more linear than the I3Ps.

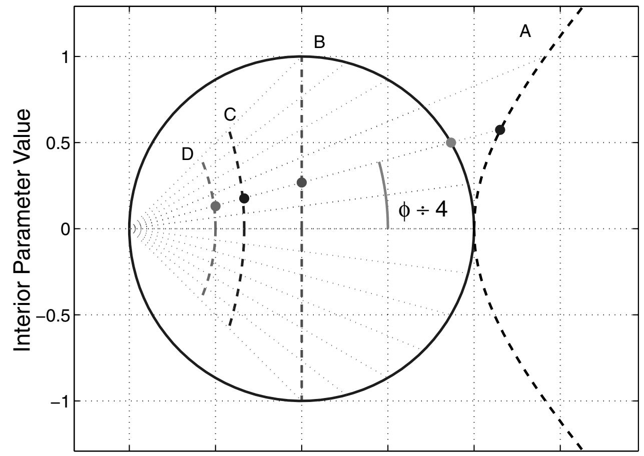

Fig. 2 A graphical display showing the increasingly linear behavior, with respect to Euler's principal angle, of some common higher-order attitude parameters: the I2Ps (B) are more linear than the CRPs (A); the I3Ps (C) are more linear than the I2Ps; and the I4Ps (D) are more linear than the I3Ps.

Singularities and Discontinuities

Every continuous three-parameter orientation set will suffer from one of two types of singularities [7]. Either the kinematic differential equations that govern their evolution will be singular or there will be particular configurations for which the parameters are undefined. The first case, commonly called a kinematic singularity, happens when an orientation generates nonunique parameter values for a continuous three-parameter set. The second case, commonly called an orientation singularity, happens when the parameters of a continuous set become undefined.

The CRPs are an example of a three-parameter set that exhibits an orientation singularity because they are undefined for rotations of 180 deg. The reciprocal parameters exhibit an orientation singularity because they are undefined at 0 deg.

The skew parameters are well defined for all orientations, but they exhibit a kinematic singularity. Kinematic singularities are revealed by, and occur at, finite values of the parameters that make $\det(\boldsymbol{B}) = 0$ or $\pm\infty$, where $\boldsymbol{B}$ is the kinematic matrix in the expression relating the relative angular velocity to the change in the attitude coordinates $\boldsymbol{\omega} = \boldsymbol{B}\dot{\boldsymbol{p}}$. For the skew parameters,

$$\det(\boldsymbol{B}) = (8 - p^2 + 4\sqrt{4 - p^2})/(\sqrt{4 - p^2})/(2 + \sqrt{4 - p^2})^3$$

which equals infinity when $p = \pm 2$. This occurs for rotations of ± 90 deg, and so in this way the skew parameters are similar to the asymmetric Euler angle sets discussed on page 455 of Shuster's work [8].

The generalized Cayley expressions here are intimately related to the higher-order Cayley expressions in [4]; the difference being that those transforms are only one way. The set of IPs (I2Ps, I3Ps, and I4Ps) that result from the generalized Cayley expressions are well defined for all orientations and have trouble-free kinematic equations (see [2] for the I2P kinematic equations and [4] for the I3P and I4P kinematic equations). This set of parameters is discontinuous, however, for orientations of 180 deg. At this specific orientation, $\varsigma = -1$ while $\boldsymbol{C} = \boldsymbol{C}^{\mathsf{T}}$. So that, rather than Eq. (9), the parameters can be directly computed from $\boldsymbol{C}$ in another way: $p_i = \pm\sqrt{\mu(1 + C_{ii})}$ for $i = 1$, 2, and 3 together with $p_i p_j = \mu C_{ij}$ for the sets $(i, j) = (1, 2)$, (2, 3), and (3, 1), where $\mu = 1/2$, $1/6$, and $(3/2 - \sqrt{2})$, respectively, for the I2Ps, I3Ps, and I4Ps.

Euler Parameters

One can choose k to be a fourth attitude parameter (i.e., p_0). But because orientation only has three degrees of freedom, the four parameters are redundant, and one must define a constraint relationship. It is convenient to select $k^2 + p^2 = p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1$. Using $\varsigma = (3k^2 - p^2)/(k^2 + p^2)$, one finds $p_0 = k(\boldsymbol{C}) = \pm\sqrt{\varsigma + 1}/2$. This choice of k together with this particular parameter constraint defines the familiar Euler parameters. Shepperd's algorithm [9] can be used to extract the Euler parameters from $\boldsymbol{C}$, and the following expressions can be used to construct $\boldsymbol{C}$ from the parameter values:

$$\boldsymbol{C} = (p_0\mathbf{1} + \boldsymbol{P})^{-1}(p_0\mathbf{1} - \boldsymbol{P}) = (p_0\mathbf{1} - \boldsymbol{P})(p_0\mathbf{1} + \boldsymbol{P})^{-1} = \mathbf{1} - 2p_0\boldsymbol{P} + 2\boldsymbol{P}\boldsymbol{P} \tag{15}$$

These equations are not new: indeed, the first set of these actually appears as equation 53 in the work of Schaub et al. [10].

Conclusions

A generalized Cayley transform relationship between orthogonal orientation matrices and skew-symmetric matrices has been introduced. This generalized transform depends on a scalar parameter that can be manipulated to generate a (Cayley) family of attitude coordinates in three-dimensional space. The corresponding relationships between orientation and attitude coordinates are bidirectionally explicit.

One class of Cayley attitude coordinates, the IPs, is special. Some characteristics of this class are that there is no orientation that the parameters cannot describe, the parameters are two-valued at orientations corresponding to 180 deg, the parameter magnitude is bounded, and the parameters are optimal in the sense that they naturally represent the shortest distance back to the origin. The IPs remove the need of shadow parameters and switching logic when parameterizing orientation.

Appendix: Orthogonal and Skew-Symmetric Matrix Relationships

A few matrix algebra manipulations are all that is needed to show that Eqs. (7) and (8) relate orthogonal and skew-symmetric matrices.

1) The equivalent expressions $k(\mathbf{1}+\boldsymbol{C})^{-1}(\mathbf{1}-\boldsymbol{C})= k(\mathbf{1}-\boldsymbol{C})(\mathbf{1}+\boldsymbol{C})^{-1}$ generate a skew-symmetric matrix when $\boldsymbol{C}$ is an orthogonal matrix; beginning with $\boldsymbol{P}=k(\mathbf{1}+\boldsymbol{C})^{-1}(\mathbf{1}-\boldsymbol{C})$, one can premultiply with $(\mathbf{1}+\boldsymbol{C})$, transpose the resulting expressions, postmultiply with $\boldsymbol{C}$, and rearrange to find $\boldsymbol{P}^{\mathsf{T}}=-\boldsymbol{P}$.

2) The equivalent expressions $(k\mathbf{1}+\boldsymbol{P})^{-1}(k\mathbf{1}-\boldsymbol{P})=(k\mathbf{1}-\boldsymbol{P})(k\mathbf{1}+\boldsymbol{P})^{-1}$ generate an orthogonal matrix when $\boldsymbol{P}$ is a skew-symmetric matrix; beginning with $\boldsymbol{C}=(k\mathbf{1}+\boldsymbol{P})^{-1}(k\mathbf{1}-\boldsymbol{P})$, one can transpose the expressions, premultiply with $\boldsymbol{C}$, and reduce to find $\boldsymbol{C}\boldsymbol{C}^{\mathsf{T}}=\mathbf{1}$.

3) Equations (7) and (8) are compatible with one another; beginning with $\boldsymbol{P}=k(\mathbf{1}+\boldsymbol{C})^{-1}(\mathbf{1}-\boldsymbol{C})$, one can premultiply with $(\mathbf{1}+\boldsymbol{C})$, distribute $\boldsymbol{P}$, and rearrange the result to isolate $\boldsymbol{C}$ to find $\boldsymbol{C}=(k\mathbf{1}-\boldsymbol{P})(k\mathbf{1}+\boldsymbol{P})^{-1}$.

The Cayley–Hamilton theorem is central to showing that Eq. (9) follows from Eq. (7), whereas Eq. (10) follows from Eq. (8).

4) Equation (9) follows from Eq. (7) via the Cayley–Hamilton theorem applied to a three-dimensional orthogonal matrix:

$$
\begin{aligned}
&\boldsymbol{C}^3-\varsigma\boldsymbol{C}^2+\varsigma\boldsymbol{C}-\mathbf{1}=\mathbf{0}\\
&\quad\Rightarrow \boldsymbol{C}\boldsymbol{C}-\varsigma\boldsymbol{C}+\varsigma\mathbf{1}-\boldsymbol{C}^{\mathsf{T}}=\mathbf{0}\\
&\quad\Rightarrow (\mathbf{1}+\boldsymbol{C})(\boldsymbol{C}^{\mathsf{T}}-\boldsymbol{C})=(\varsigma+1)(\mathbf{1}-\boldsymbol{C})\\
&\quad\Rightarrow \frac{k}{\varsigma+1}(\boldsymbol{C}^{\mathsf{T}}-\boldsymbol{C})=k(\mathbf{1}+\boldsymbol{C})^{-1}(\mathbf{1}-\boldsymbol{C})\\
&\quad\Rightarrow \boldsymbol{P}=\frac{k}{\varsigma+1}(\boldsymbol{C}^{\mathsf{T}}-\boldsymbol{C})
\end{aligned}
$$

5) The derivation of Eq. (10) can begin with adding and subtracting quantities from $k\mathbf{1}-\boldsymbol{P}$:

$$k\mathbf{1}-\boldsymbol{P}=k\mathbf{1}+\boldsymbol{P}-2\boldsymbol{P}=k\mathbf{1}+\boldsymbol{P}+\frac{2}{p^2+k^2}(-p^2\boldsymbol{P}-k^2\boldsymbol{P})$$

At this point, the Cayley–Hamilton theorem applied to a three-dimensional skew-symmetric matrix $\boldsymbol{P}^3+p^2\boldsymbol{P}=\mathbf{0}$ can be used:

$$k\mathbf{1}-\boldsymbol{P}=k\mathbf{1}+\boldsymbol{P}+\frac{2}{p^2+k^2}(\boldsymbol{P}^3-k^2\boldsymbol{P})$$

The right side can be written as the result of a matrix product. This allows Eq. (8) to be used, which finally leads to Eq. (10):

$$
\begin{aligned}
&k\mathbf{1}-\boldsymbol{P}=(k\mathbf{1}+\boldsymbol{P})\left(\mathbf{1}-\frac{2k}{p^2+k^2}\boldsymbol{P}+\frac{2}{p^2+k^2}\boldsymbol{P}\boldsymbol{P}\right)\\
&\quad\Rightarrow (k\mathbf{1}+\boldsymbol{P})^{-1}(k\mathbf{1}-\boldsymbol{P})=\mathbf{1}-\frac{2k}{p^2+k^2}\boldsymbol{P}+\frac{2}{p^2+k^2}\boldsymbol{P}\boldsymbol{P}\\
&\quad\Rightarrow \boldsymbol{C}=\mathbf{1}-\frac{2k}{p^2+k^2}\boldsymbol{P}+\frac{2}{p^2+k^2}\boldsymbol{P}\boldsymbol{P}
\end{aligned}
$$

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